# A Conductive Wedge in Yee's Mesh

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***Abstract*— This letter presents a correction of the finite-difference (FD) methods employing Yee's mesh, improving the accuracy when conductive edge corners are present in the analyzed structure. The algorithm allows the wedge to be arbitrarily located with respect to the grid points and yields significant error reduction.**

***Index Terms*— FDFD and FDTD methods, finite-difference methods.**

## I. Introduction

Finite-difference (FD) schemes using Yee's mesh are very attractive for modeling fields in frequency [1] and time [2] domain. One of the assumptions underlying these schemes is that the field variation between cells is linear. This assumption is not satisfied in the vicinity of the conductive wedges, where singularities occur [3]. These singularities cause the local approximation error to be unbounded and may produce significant global inaccuracies. To avoid this problem without increasing the mesh density, one may use a technique known as a method of Motz and Woods [4], which locally modifies the scheme to incorporate the analytical formulae expressing the field behavior in the neighborhood of the wedge. Mur [5] and several authors [6], [7] adapted this technique for Yee's mesh, but published solutions used only the lowest order approximation and assumed that the edge is located centrally within Yee's cell. In this letter we present the higher order approximation algorithm which enables one to place the edge at any position within a cell. Our algorithm is based on the integral form of Maxwell's equations and may be regarded as an extension of Mur's scheme.

## II. Local Modification

Let us consider a two-dimensional (2-D) conductive wedge placed in Yee's mesh [Fig. 1(a)]. The $z$ field components in the vicinity of the wedge can be represented by the following series:

$$\begin{aligned} E_z(r,\phi,\xi) &= c_1^e(\xi) r^{\nu_1}\sin(\nu_1\phi) + c_2^e(\xi) r^{\nu_2}\sin(\nu_2\phi) + \cdots \\ H_z(r,\phi,\xi) &= c_0^h(\xi) + c_1^h(\xi) r^{\nu_1}\cos(\nu_1\phi) \\ &\quad + c_2^h(\xi) r^{\nu_2}\cos(\nu_2\phi) + \cdots \end{aligned} \tag{1}$$

where $r$ and $\phi$ denote local polar coordinates, $\xi$ represents time or frequency, and $\nu_k = k\pi/(2\pi - \alpha)$ where $\alpha$ is the edge angle. Expansion coefficients are not known *a priori* but can be determined from known values of the fields at the grid points located in the vicinity of the wedge. For concreteness let us consider the finite-difference time-domain (FDTD) analysis of the TM field near the knife edge corner $\alpha = 0, \nu_k = k/2$. (The derivation is analogous for the FDFD and edge angles $\alpha$ other than zero.) The arrangement of the mesh is shown in Fig. 1(b). To calculate the expansion coefficients we use three points with local coordinates $(r_i, \phi_i)$, denoted in the figure as $E_{zi}, i = 1,2,3$. Equation (1) at points $(r_i, \phi_i)$ may be represented in the matrix form as

$$\underline{E}_z = \underline{\underline{B}}_e \underline{c}^e \tag{2}$$

where $\underline{\underline{B}}_e$ is a $3 \times 3$ matrix with the elements

$$b_{eik} = r_i^{k/2}\sin(k\phi_i/2). \tag{3}$$

To account for an arbitrary position of the corner within a cell, we next use the integral form of Maxwell's equations. In the integral approach average values of the $H$ field (for the TM polarization) along the cell edges are required in the FDTD to update the $E$ field. These average values are calculated by integrating the expression for curl of the $E$ field along grid edges. In the vicinity of the edge the curl of $E_z$ field is computed using series (1). Accordingly, the average value of the time derivative for $H_{y1}$ field is given by

$$\frac{d}{dt}H_{y1} = \sum_{k=1}^{3} \frac{c_k^e(t)k}{2\mu\Delta y}\int_{y_1}^{y_1+\Delta y} \sin[(k/2-1)\phi] r^{k/2-1}\, dy \tag{4}$$

where $y_1$ denotes the $y$ coordinate of the lower right corner of the shaded cell in Fig. 1(b) and $\Delta y$ is the mesh size in the $y$ direction. A similar formula can be written for the $H_{x1}$ and $H_{x2}$ components. Since the coefficients $c_k^e(t)$ are calculated from fields $E_{zi}(\underline{c}^e = \underline{\underline{B}}_e^{-1}\underline{E}_z)$, the formula for the time derivative of the $H$ field for the three points in the vicinity of the edge can be written in the form

$$\frac{d}{dt}\underline{H}_t = \underline{\underline{A}}_e \underline{\underline{B}}_e^{-1}\underline{E}_z \tag{5}$$

with rows $\underline{\underline{A}}_e$ being the matrix representations of line integrals along cell edges [cf. (4)] and vectors $\underline{E}$ and $\underline{H}$ consisting of $E_{z1}, E_{z2}, E_{z3}$ and $H_{x1}, H_{x2}, H_{y1}$, respectively. Likewise, for the TE polarization [Fig. 1(c)] series (1) is used and coefficients $c_i^h$ are computed based on the relation $\underline{\underline{B}}_h^{-1}\underline{H}_z = \underline{c}^h$ with vector $\underline{H}_z$ consisting of fields $H_{z1}, H_{z2}, H_{z3}, H_{z4}$ and

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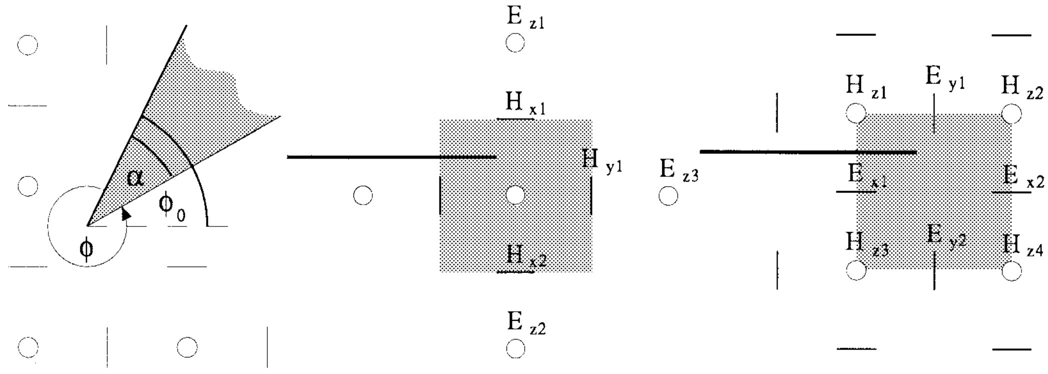
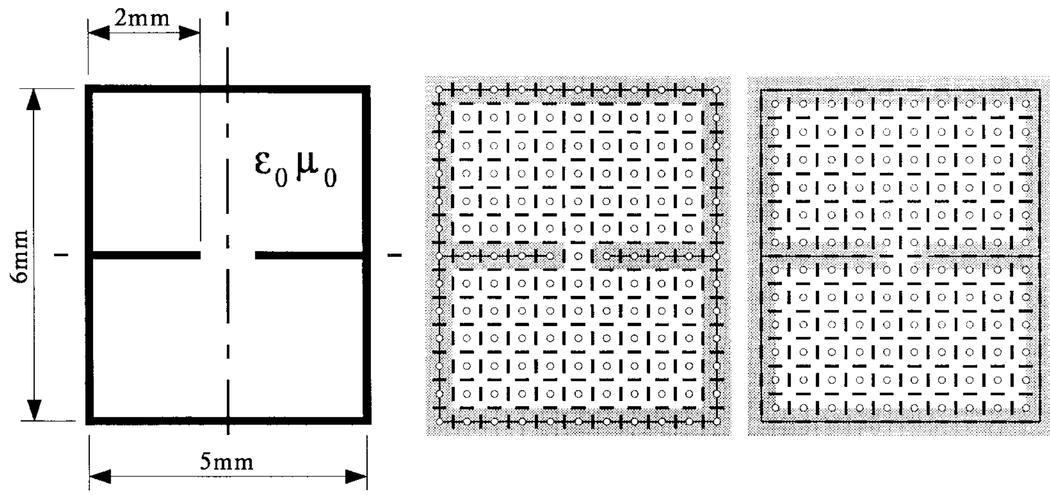



(a) (b) (c)

Fig. 1. Conductive wedge placed in Yee's mesh: (a) general case and arrangement of the mesh for $\alpha = 0, \phi_0 = \pi$ for (b) TM and (c) TE polarization.




(a) (b) (c)

Fig. 2. Test model of: (a) the fin line and FD grids, (b) TM and, (c) TE polarization. Grey area in (b) and (c) indicates possible location of metal boundaries with respect to Yee's grid.

the elements of a 4 × 4 matrix $\underline{\underline{B}}_h$ given by

$$b_{hik} = r_i^{(k-1)/2} \cos[(k-1)\phi_i/2]. \tag{6}$$

Average values of $E_{x1}, E_{x2}, E_{y1},$ and $E_{y2}$ are calculated by taking the curl of $H_z$ computed using series expansion and integration along relevant cell edges. The local formula is then

$$\frac{d}{dt}\underline{E}_t = \underline{\underline{A}}_h \underline{\underline{B}}_h^{-1} \underline{H}_z \tag{7}$$

with rows of matrix $\underline{\underline{A}}_h$ representing line integrals of the series expansion of curl $E_z$ along four cell edges.

## III. Numerical Results

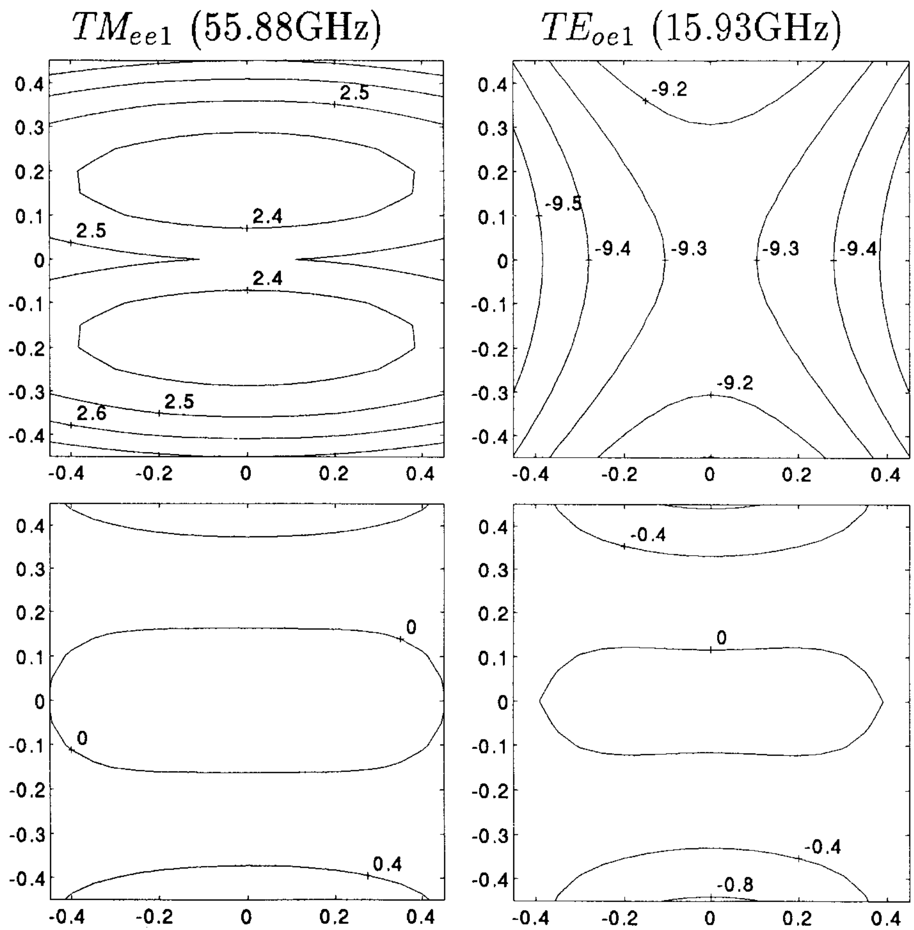
We implemented the new algorithm in the FDTD code and computed the cutoff frequencies of a fin line shown in Fig. 2. The eigenvalue analysis shows that the scheme is stable. Table I compares the results for the $\mathrm{TM}_{ee1}$ and $\mathrm{TE}_{oe1}$ modes for the standard and modified FDTD algorithm when the knife edge corner is located in the center of Yee's cell. Cutoff frequencies obtained from the Richardson extrapolation of the results obtained by a standard FDTD for a sequence of grids with increasing density were used as a reference.

TABLE I
Cutoff Frequencies in Gigahertz and Relative Errors for the Standard and New FDTD Algorithm

| | Standard | | | | New |
|---|---|---|---|---|---|
| | 10×12 | 40×48 | 160×192 | Ext. | 10×12 |
| $\mathrm{TM}_{ee1}$ | 57.27<br>2.50% | 56.30<br>0.75% | 55.99<br>0.19% | 55.88 | 55.83<br>-0.09% |
| $\mathrm{TE}_{oe1}$ | 14.45<br>-9.28% | 15.57<br>-2.23% | 15.84<br>-0.55% | 15.93 | 15.93<br>0.04% |
| $\mathrm{TE}_{ee1}$ | 30.24<br>-1.68% | 30.63<br>-0.43% | 30.73<br>-0.11% | 30.76 | 30.73<br>-0.11% |

In a standard FDTD algorithm the error for the coarsest grid 10 × 12 reaches 9.28% for the $\mathrm{TE}_{oe1}$ mode and drops below 1% only when the grid resolution is increased by the factor of 8 in each direction. When the new algorithm is applied, even the coarsest grid is sufficient to obtain accuracy below 0.1%.

To illustrate the ability of the new algorithm to deal with an arbitrary location of the corner within Yee's cell we carried out a series of simulations for the same structure but the 10 × 12-




(a) (b)

Fig. 3. Relative error [%] of the cutoff frequencies versus the normalized wedge location within Yee's cell for the algorithm (a) without and (b) with correction.

grid translated in the $x, y$ space by the normalized distance $-0.45 \leq x_t/\Delta x \leq 0.45, -0.45 \leq y_t/\Delta y \leq 0.45$. The contour plots showing the relative error for the algorithm with and without correction are given in Fig. 3. For the standard method the largest error is at the level of 9% $(\mathrm{TE}_{oe1}$ mode) or 2.5% $(\mathrm{TM}_{oe1})$ for all locations of edge within Yee's cell while the modified algorithm reduces this error to 0.4% and 0.8% in the worst case.

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